Lomonosov Moscow State University
Faculty of Mechanics and Mathematics
Entrance exam for the Master's degree program
"Geometry and Quantum Fields" (written part)
July 2023

Examination card no. 2023-07-25-e

1. Find the following limit:

$$\lim_{x\to 0} \frac{6\sin(x^2) - 6x^2 + x^6}{x\left(e^{x^3} - 1 - x^3 - \frac{1}{2}x^6\right)}$$

2. Evaluate the following indefinite integral:

$$\int \frac{1}{x^3 + 2x^2 + x}\, dx$$

3. Find the electric field generated by two parallel infinite metal planes. The distance between the planes is l and both planes carry electric charge. The charge densities are constant for both planes and are given respectively by σ_1 and σ_2.

4. A spatial pendulum with a bob of mass m is suspended from a fixed point by a massless rigid rod of length l. Suppose that the system is in a uniform gravitational field.

   a) Determine the Lagrangian of the system in spherical coordinates.

   b) Identify two independent conserved quantities.

5. Consider the following matrix:

$$\begin{pmatrix} 7 & 14 & -13 \\ 1 & 2 & -1 \\ 6 & 12 & -12 \end{pmatrix}$$

   as the matrix of a liner operator on the 3-dimensional real Euclidean space with a fixed orthonormal basis. Find $\sin(\alpha)$, where α is the angle between the eigenvectors with the largest and smallest eigenvalue (seen as real numbers).

6. Find the general solution $y(x)$ to the following equation:

$$y'' + \alpha^2 y + \sin(\beta t) = 0\,, \qquad \alpha, \beta \in \mathbb{R}.$$

   What physical system is described by this equation?

7. Consider a 3 dimensional real Euclidean space with standard coordinates x, y, z. Consider the line determined by $\frac{x-8}{7} = \frac{y-2}{1} = \frac{z}{2}$ and the plane determined by $3x + 2y + 5z + 5 = 0$. The line reflects in the plane in a standard way (as a light ray in the flat mirror). Find the equation determining the reflected line

8. Determine the energy spectrum of a quantum system whose classical limit is described by the following Lagrangian:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\alpha^2}{2}(5x^2 - 4xy + 5y^2)\,.$$