Admission examination

Lomonosov Moscow State University

«Geometry and quantum fields»

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1. Find the following limit:
$$\lim_{x\to 0}\frac{e^{-x}-1+x-x^2}{1-\cos 5x}.$$

2. Evaluate the following indefinite integral:
$$\int x\,\sin^3 x^2\,\cos^3 x^2\,dx.$$

3. Let x^i, $i=1,2,3$ be coordinates on $\mathbb{R}^3$ and $V_i(x)$ a vector-valued function. Find the general solution to the following differential equation: $\frac{\partial}{\partial x^i}V_j(x)+\frac{\partial}{\partial x^j}V_i(x)=0$. What is the geometric interpretation of the solution?

4. A spatial pendulum with a bob of mass m is suspended from a fixed point by a massless rigid rod of length l. Suppose that the system is in a uniform gravitational field.

   a) Determine the Lagrangian of the system in spherical coordinates.

   b) Identify two independent conserved quantities.

5. Consider the following matrix:
$$\begin{pmatrix} -11 & -34 & -26 \\ 8 & 25 & 20 \\ -4 & -14 & -13 \end{pmatrix}$$
   as the matrix of a liner operator on the 3-dimensional real Euclidean space with a fixed orthonormal basis. Find $\sin(\alpha)$, where α is the angle between the eigenvectors with the largest and smallest eigenvalue (seen as real numbers).

6. Find the general solution $y(x)$ to the following equation:
$$y''+\alpha^2 y+\sin(\beta t)=0\,,\qquad \alpha,\beta\in\mathbb{R}.$$
   What physical system is described by this equation?

7. Consider a convex polygone with N vertexes. All the vertexes are connected by a random broken line consisting of $(N-1)$ segments. What is the probability that the broken line intersects itself at least once.

8. Determine the energy spectrum of a quantum system whose classical limit is described by the following Lagrangian:
$$L(x,y,\dot{x},\dot{y})=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{\alpha^2}{2}(7x^2-2xy+7y^2)\,.$$